


Energy is the capacity to do work.

- Radiant energy comes from the sun and is earth's primary energy source
- Thermal energy is the energy associated with the random motion of atoms and molecules
- Chemical energy is the energy stored within the bonds of chemical substances
- Nuclear energy is the energy stored within the collection of neutrons and protons in the atom
- Potential energy is the energy available by virtue of an object's position


## Energy Changes in Chemical Reactions

Heat is the transfer of thermal energy between two bodies that are at different temperatures.

Temperature is a measure of the thermal energy.

Temperature $*$ Thermal Energy


Thermochemistry is the study of heat change in chemical reactions.

The system is the specific part of the universe that is of interest in the study.


Exothermic process is any process that gives off heat transfers thermal energy from the system to the surroundings.

$$
\begin{gathered}
2 \mathrm{H}_{2}(\mathrm{~g})+\mathrm{O}_{2}(\mathrm{~g}) \longrightarrow 2 \mathrm{H}_{2} \mathrm{O}(\mathrm{l})+\text { energy } \\
\mathrm{H}_{2} \mathrm{O}(\mathrm{~g}) \longrightarrow \mathrm{H}_{2} \mathrm{O}(\mathrm{l})+\text { energy }
\end{gathered}
$$

Endothermic process is any process in which heat has to be supplied to the system from the surroundings.

$$
\begin{gathered}
\text { energy }+2 \mathrm{HgO}(\mathrm{~s}) \longrightarrow 2 \mathrm{Hg}(\mathrm{l})+\mathrm{O}_{2}(\mathrm{~g}) \\
\text { energy }+\mathrm{H}_{2} \mathrm{O}(\mathrm{~s}) \longrightarrow \mathrm{H}_{2} \mathrm{O}(()
\end{gathered}
$$

Thermodynamics is the scientific study of the interconversion of heat and other kinds of energy.

State functions are properties that are determined by the state of the system, regardless of how that condition was achieved. energy, pressure, volume, temperature


$$
\begin{aligned}
\Delta U & =U_{\text {final }}-U_{\text {initial }} \\
\Delta P & =P_{\text {final }}-P_{\text {initial }} \\
\Delta V & =V_{\text {final }}-V_{\text {initial }} \\
\Delta T & =T_{\text {final }}-T_{\text {initial }}
\end{aligned}
$$

Potential energy of hiker 1 and hiker 2 is the same even though they took different paths.

## First law of thermodynamics - energy can be

 converted from one form to another, but cannot be created or destroyed.$$
\begin{gathered}
\Delta U_{\text {system }}+\Delta U_{\text {surroundings }}=0 \\
\text { or }
\end{gathered}
$$

$$
\Delta U_{\text {system }}=-\Delta U_{\text {surroundings }}
$$

$$
\mathrm{C}_{3} \mathrm{H}_{8}+5 \mathrm{O}_{2} \longrightarrow 3 \mathrm{CO}_{2}+4 \mathrm{H}_{2} \mathrm{O}
$$

Exothermic chemical reaction!

Chemical energy lost by combustion = Energy gained by the surroundings system surroundings

## Another form of the first law for $\Delta U_{\text {system }}$

$\Delta U=q+w$
$\Delta U$ is the change in internal energy of a system
$q$ is the heat exchange between the system and the surroundings $w$ is the work done on (or by) the system
$w=-P \Delta V$ when a gas expands against a constant external pressure
table 6.1 Sign Conventions for Work and Heat

Work done by the system on the surroundings
Work done on the system by the surroundings
Heat absorbed by the system from the surroundings (endothermic process)
Heat absorbed by the surroundings from the system (exothermic process)

## Work Done On the System

$$
\begin{aligned}
& w=F \times d \\
& w=-P \Delta V
\end{aligned}
$$

$$
P \times V=\frac{F}{d^{2}} \times d^{3}=F \times d=w \quad-P V>0
$$

$$
w_{s y s}<0
$$

Work is not a state function.

$$
\Delta w \equiv w_{\text {final }}-w_{\text {initial }}
$$


initial
final

A sample of nitrogen gas expands in volume from 1.6 L to 5.4 L at constant temperature. What is the work done in joules if the gas expands (a) against a vacuum and (b) against a constant pressure of 3.7 atm ?

$$
w=-P \Delta V
$$

(a) $\quad \Delta V=5.4 \mathrm{~L}-1.6 \mathrm{~L}=3.8 \mathrm{~L} \quad P=0 \mathrm{~atm}$

$$
W=-0 \mathrm{~atm} \times 3.8 \mathrm{~L}=0 \mathrm{~L} \cdot \mathrm{~atm}=0 \text { joules }
$$

(b) $\quad \Delta V=5.4 \mathrm{~L}-1.6 \mathrm{~L}=3.8 \mathrm{~L} \quad P=3.7 \mathrm{~atm}$

$$
\begin{aligned}
& w=-3.7 \mathrm{~atm} \times 3.8 \mathrm{~L}=-14.1 \mathrm{~L} \cdot \mathrm{~atm} \\
& w=-14.1 \mathrm{~L} \cdot \mathrm{~atm} \times \frac{101.3 \mathrm{~J}}{1 \mathrm{~L} \cdot \mathrm{~atm}}=-1430 \mathrm{~J}
\end{aligned}
$$

## Enthalpy and the First Law of Thermodynamics

$$
\Delta U=q+w
$$

At constant pressure:

$$
q=\Delta H \text { and } w=-P \Delta V
$$

$$
\Delta E=\Delta H-P \Delta V
$$

$$
\Delta H=\Delta U+P \Delta V
$$



Enthalpy $(H)$ is used to quantify the heat flow into or out of a system in a process that occurs at constant pressure.

$$
\Delta H=H \text { (products) }-H \text { (reactants) }
$$

$\Delta H=$ heat given off or absorbed during a reaction at constant pressure

$H_{\text {products }}<H_{\text {reactants }}$
$\Delta H<0$

$H_{\text {products }}>H_{\text {reactants }}$ $\Delta H>0$

## Thermochemical Equations

 Is $\Delta H$ negative or positive?

System absorbs heat
Endothermic
$\Delta H>0$
6.01 kJ are absorbed for every 1 mole of ice that melts at $0^{\circ} \mathrm{C}$ and 1 atm .

$$
\mathrm{H}_{2} \mathrm{O}(\mathrm{~s}) \longrightarrow \mathrm{H}_{2} \mathrm{O}(l) \quad \Delta H=6.01 \mathrm{~kJ} / \mathrm{mol}
$$

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## Thermochemical Equations



Is $\Delta H$ negative or positive?

System gives off heat
Exothermic
$\Delta H<0$
890.4 kJ are released for every 1 mole of methane that is combusted at $25^{\circ} \mathrm{C}$ and 1 atm .

$$
\mathrm{CH}_{4}(g)+2 \mathrm{O}_{2}(\mathrm{~g}) \longrightarrow \mathrm{CO}_{2}(\mathrm{~g})+2 \mathrm{H}_{2} \mathrm{O}(\mathrm{I}) \quad \Delta H=-890.4 \mathrm{~kJ} / \mathrm{mol}
$$

## Thermochemical Equations

- The stoichiometric coefficients always refer to the number of moles of a substance

$$
\mathrm{H}_{2} \mathrm{O}(s) \longrightarrow \mathrm{H}_{2} \mathrm{O}(l) \quad \Delta H=6.01 \mathrm{~kJ} / \mathrm{mol}
$$

- If you reverse a reaction, the sign of $\Delta H$ changes

$$
\mathrm{H}_{2} \text { (II) } \longrightarrow \mathrm{H}_{2} \mathrm{O}(\mathrm{~s}) \quad \Delta H=-6.01 \mathrm{~kJ} / \mathrm{mol}
$$

- If you multiply both sides of the equation by a factor $n$, then $\Delta H$ must change by the same factor $n$.

$$
2 \mathrm{H}_{2} \mathrm{O}(s) \longrightarrow 2 \mathrm{H}_{2} \mathrm{O}(l) \quad \Delta H=2 \times 6.01=12.0 \mathrm{~kJ}
$$

## Thermochemical Equations

- The physical states of all reactants and products must be specified in thermochemical equations.

$$
\begin{array}{ll}
\mathrm{H}_{2} \mathrm{O}(\mathrm{~s}) \longrightarrow \mathrm{H}_{2} \mathrm{O}(\mathrm{II}) & \Delta H=6.01 \mathrm{~kJ} / \mathrm{mol} \\
\mathrm{H}_{2} \text { (II) } \longrightarrow \mathrm{H}_{2} \mathrm{O}(\mathrm{~g}) & \Delta H=44.0 \mathrm{~kJ} / \mathrm{mol}
\end{array}
$$

How much heat is evolved when 266 g of white phosphorus $\left(\mathrm{P}_{4}\right)$ burn in air?

$$
\begin{aligned}
& \mathrm{P}_{4}(\mathrm{~s})+5 \mathrm{O}_{2}(g) \longrightarrow \mathrm{P}_{4} \mathrm{O}_{10}(\mathrm{~s}) \quad \Delta H=-3013 \mathrm{~kJ} / \mathrm{mol} \\
& 266 g P_{4} \times \frac{1 \mathrm{~mol}_{4}}{123.9 g P_{4}} \times \frac{3013 \mathrm{~kJ}}{1 \mathrm{mof} \mathrm{P}_{4}}=6470 \mathrm{~kJ}
\end{aligned}
$$

## A Comparison of $\Delta H$ and $\Delta U$

$$
\begin{gathered}
2 \mathrm{Na}(s)+2 \mathrm{H}_{2} \mathrm{O}(I) \longrightarrow 2 \mathrm{NaOH}(a q)+\mathrm{H}_{2}(g) \Delta \mathrm{H}=-367.5 \mathrm{~kJ} / \mathrm{mol} \\
\Delta U=\Delta H-P \Delta V \quad \text { At } 25^{\circ} \mathrm{C}, 1 \text { mole } \mathrm{H}_{2}=24.5 \mathrm{~L} \text { at } 1 \mathrm{~atm}
\end{gathered}
$$

$P \Delta V=1$ atm $\times 24.5 \mathrm{~L}=2.5 \mathrm{~kJ}$

$$
\Delta U=-367.5 \mathrm{~kJ} / \mathrm{mol}-2.5 \mathrm{~kJ} / \mathrm{mol}=-370.0 \mathrm{~kJ} / \mathrm{mol}
$$



The specific heat (s) of a substance is the amount of heat (q) required to raise the temperature of one gram of the substance by one degree Celsius.

The heat capacity $(\boldsymbol{C})$ of a substance is the amount of heat $(q)$ required to raise the temperature of a given quantity $(m)$ of the substance by one degree Celsius.

| The Specific Heats of Some Common Substances |  |
| :---: | :---: |
| Substance | Specific Heat $\left(\mathrm{J} / \mathrm{g} \cdot{ }^{\circ} \mathrm{C}\right)$ |
| Al | 0.900 |
| Au | 0.129 |
| C (graphite) | 0.720 |
| C (diamond) | 0.502 |
| Cu | 0.385 |
| Fe | 0.444 |
| Hg | 0.139 |
| $\mathrm{H}_{2} \mathrm{O}$ | 4.184 |
| $\mathrm{C}_{2} \mathrm{H}_{5} \mathrm{OH}$ (ethanol) | 2.46 |

$$
C=m \times s
$$

Heat (q) absorbed or released:

$$
\begin{gathered}
q=m \times s \times \Delta t \\
q=C \times \Delta t \\
\Delta t=t_{\text {final }}-t_{\text {initial }}
\end{gathered}
$$

$\mathrm{C}_{2} \mathrm{H}_{5} \mathrm{OH}$ (ethanol) 2.46

How much heat is given off when an 869 g iron bar cools from $94^{\circ} \mathrm{C}$ to $5^{\circ} \mathrm{C}$ ?

$$
s \text { of } \mathrm{Fe}=0.444 \mathrm{~J} / \mathrm{g} \cdot{ }^{\circ} \mathrm{C}
$$

$\Delta t=t_{\text {final }}-t_{\text {initial }}=5^{\circ} \mathrm{C}-94^{\circ} \mathrm{C}=-89^{\circ} \mathrm{C}$
$q=m s \Delta t=869 g \times 0.444 \mathrm{~J} / g \cdot{ }^{\circ} \mathscr{C}^{\prime} \times-89^{\circ} \mathcal{C}=-34,000 \mathrm{~J}$


## Constant-Volume Calorimetry



No heat enters or leaves!
$q_{\text {sys }}=q_{\text {water }}+q_{\text {bomb }}+q_{\mathrm{rxn}}$
$q_{\text {sys }}=0$
$q_{\text {rxn }}=-\left(q_{\text {water }}+q_{\text {bomb }}\right)$
$q_{\text {water }}=m \times s \times \Delta t$
$q_{\text {bomb }}=C_{\text {bomb }} x \Delta t$
Reaction at Constant $V$
$\Delta H \neq q_{\mathrm{rxn}}$
$\Delta H \sim q_{\mathrm{rxn}}$

## Constant-Pressure Calorimetry


$q_{\text {sys }}=q_{\text {water }}+q_{\mathrm{cal}}+q_{\mathrm{rxn}}$
$q_{\text {sys }}=0$
$q_{\mathrm{rxn}}=-\left(q_{\text {water }}+q_{\text {cal }}\right)$
$q_{\text {water }}=m \times \mathrm{s} \times \Delta t$
$q_{\text {cal }}=C_{\text {cal }} x \Delta t$
Reaction at Constant $P$
$\Delta H=q_{\text {rx }}$
No heat enters or leaves!

TABLE 6.3 Heats of Some Typical Reactions Measured at Constant Pressure

| Type of <br> Reaction | Example | $\Delta \boldsymbol{H}$ <br> (kJ/mol) |
| :--- | :--- | ---: |
| Heat of neutralization | $\mathrm{HCl}(a q)+\mathrm{NaOH}(a q) \longrightarrow \mathrm{NaCl}(a q)+\mathrm{H}_{2} \mathrm{O}(l)$ | -56.2 |
| Heat of ionization | $\mathrm{H}_{2} \mathrm{O}(l) \longrightarrow \mathrm{H}^{+}(a q)+\mathrm{OH}^{-}(a q)$ | 56.2 |
| Heat of fusion | $\mathrm{H}_{2} \mathrm{O}(s) \longrightarrow \mathrm{H}_{2} \mathrm{O}(l)$ | 6.01 |
| Heat of vaporization | $\mathrm{H}_{2} \mathrm{O}(l) \longrightarrow \mathrm{H}_{2} \mathrm{O}(g)$ | $44.0^{*}$ |
| Heat of reaction | $\mathrm{MgCl}_{2}(s)+2 \mathrm{Na}(l) \longrightarrow 2 \mathrm{NaCl}(s)+\mathrm{Mg}(s)$ | -180.2 |

Measured at $25^{\circ} \mathrm{C}$. At $100^{\circ} \mathrm{C}$, the value is 40.79 kJ .

Because there is no way to measure the absolute value of the enthalpy of a substance, must I measure the enthalpy change for every reaction of interest?

Establish an arbitrary scale with the standard enthalpy of formation $\left(\Delta H_{f}^{\circ}\right)$ as a reference point for all enthalpy expressions.

Standard enthalpy of formation $\left(\Delta H_{f}^{\circ}\right)$ is the heat change that results when one mole of a compound is formed from its elements at a pressure of 1 atm .

The standard enthalpy of formation of any element in its most stable form is zero.

$$
\begin{array}{cc}
\Delta H_{\mathrm{f}}^{\circ}\left(\mathrm{O}_{2}\right)=0 & \Delta H_{\mathrm{f}}^{\circ}(\mathrm{C}, \text { graphite })=0 \\
\Delta H_{\mathrm{f}}^{\circ}\left(\mathrm{O}_{3}\right)=142 \mathrm{~kJ} / \mathrm{mol} & \Delta H_{\mathrm{f}}^{\circ}(\mathrm{C}, \text { diamond })=1.90 \mathrm{~kJ} / \mathrm{mol}
\end{array}
$$

TABLE 6.4 Standard Enthalpies of Formation of Some Inorganic

| Substance | $\Delta H_{\mathrm{f}}^{\mathrm{o}}(\mathrm{kJ} / \mathrm{mol})$ | Substance | $\Delta H_{f}^{\circ}(\mathrm{kJ} / \mathrm{mol})$ |
| :---: | :---: | :---: | :---: |
| $\mathrm{Ag}(\mathrm{s})$ | 0 | $\mathrm{H}_{2} \mathrm{O}_{2}(l)$ | -187.6 |
| $\mathrm{AgCl}(\mathrm{s})$ | -127.0 | $\mathrm{Hg}(l)$ | 0 |
| $\mathrm{Al}(\mathrm{s})$ | 0 | $\mathrm{I}_{2}(s)$ | 0 |
| $\mathrm{Al}_{2} \mathrm{O}_{3}(\mathrm{~s})$ | -1669.8 | $\mathrm{HI}(\mathrm{g})$ | 25.9 |
| $\mathrm{Br}_{2}(l)$ | 0 | $\mathrm{Mg}(\mathrm{s})$ | 0 |
| $\mathrm{HBr}(\mathrm{g})$ | -36.2 | $\mathrm{MgO}(\mathrm{s})$ | -601.8 |
| C(graphite) | 0 | $\mathrm{MgCO}_{3}(\mathrm{~s})$ | -1112.9 |
| C(diamond) | 1.90 | $\mathrm{N}_{2}(\mathrm{~g})$ | 0 |
| $\mathrm{CO}(\mathrm{g})$ | $-110.5$ | $\mathrm{NH}_{3}(\mathrm{~g})$ | -46.3 |
| $\mathrm{CO}_{2}(\mathrm{~g})$ | -393.5 | $\mathrm{NO}(g)$ | 90.4 |
| $\mathrm{Ca}(\mathrm{s})$ | 0 | $\mathrm{NO}_{2}(\mathrm{~g})$ | 33.85 |
| $\mathrm{CaO}(s)$ | -635.6 | $\mathrm{N}_{2} \mathrm{O}(\mathrm{g})$ | 81.56 |
| $\mathrm{CaCO}_{3}(s)$ | -1206.9 | $\mathrm{N}_{2} \mathrm{O}_{4}(\mathrm{~g})$ | 9.66 |
| $\mathrm{Cl}_{2}(\mathrm{~g})$ | 0 | $\mathrm{O}(\mathrm{g})$ | 249.4 |
| $\mathrm{HCl}(\mathrm{g})$ | -92.3 | $\mathrm{O}_{2}(\mathrm{~g})$ | 0 |
| $\mathrm{Cu}(\mathrm{s})$ | 0 | $\mathrm{O}_{3}(\mathrm{~g})$ | 142.2 |
| $\mathrm{CuO}(\mathrm{s})$ | -155.2 | S (rhombic) | 0 |
| $\mathrm{F}_{2}(\mathrm{~g})$ | 0 | S (monoclinic) | 0.30 |
| $\mathrm{HF}(\mathrm{g})$ | -271.6 | $\mathrm{SO}_{2}(\mathrm{~g})$ | -296.1 |
| $\mathrm{H}(\mathrm{g})$ | 218.2 | $\mathrm{SO}_{3}(\mathrm{~g})$ | -395.2 |
| $\mathrm{H}_{2}(\mathrm{~g})$ | 0 | $\mathrm{H}_{2} \mathrm{~S}(\mathrm{~g})$ | -20.15 |
| $\mathrm{H}_{2} \mathrm{O}(\mathrm{g})$ | -241.8 | $\mathrm{Zn}(\mathrm{s})$ | 0 |
| $\mathrm{H}_{2} \mathrm{O}(l)$ | -285.8 | $\mathrm{ZnO}(\mathrm{s})$ | -348.0 |

The standard enthalpy of reaction $\left(\Delta \mathrm{H}_{\mathrm{rxn}}^{\circ}\right)$ is the enthalpy of a reaction carried out at 1 atm .

$$
\begin{gathered}
a \mathrm{~A}+b \mathrm{~B} \longrightarrow c \mathrm{C}+d \mathrm{D} \\
\Delta H_{\mathrm{rxn}}^{\circ}=\left[c \Delta H_{\mathrm{f}}^{\circ}(\mathrm{C})+d \Delta H_{\mathrm{f}}^{\circ}(\mathrm{D})\right]-\left[a \Delta H_{\mathrm{f}}^{\circ}(\mathrm{A})+b \Delta H_{\mathrm{f}}^{\circ}(\mathrm{B})\right] \\
\Delta H_{\mathrm{rxn}}^{\circ}=\Sigma n \Delta H_{\mathrm{f}}^{\circ}(\text { products })-\Sigma m \Delta H_{\mathrm{f}}^{\circ} \text { (reactants) }
\end{gathered}
$$

Hess's Law: When reactants are converted to products, the change in enthalpy is the same whether the reaction takes place in one step or in a series of steps.
(Enthalpy is a state function. It doesn' t matter how you get there, only where you start and end.)


Calculate the standard enthalpy of formation of $\mathrm{CS}_{2}(I)$ given that:
C (graphite) $+\mathrm{O}_{2}(\mathrm{~g}) \longrightarrow \mathrm{CO}_{2}(\mathrm{~g}) \quad \Delta H_{\mathrm{rxn}}^{\circ}=-393.5 \mathrm{~kJ} / \mathrm{mol}$
S (rhombic) $+\mathrm{O}_{2}(\mathrm{~g}) \longrightarrow \mathrm{SO}_{2}(\mathrm{~g}) \quad \Delta H^{\rho} \mathrm{rxF}-296.1 \mathrm{~kJ} / \mathrm{mol}$
$\mathrm{CS}_{2}(\mathrm{l})+3 \mathrm{O}_{2}(\mathrm{~g}) \longrightarrow \mathrm{CO}_{2}(\mathrm{~g})+2 \mathrm{SO}_{2}(\mathrm{~g}) \quad \Delta H^{\circ} \underset{\mathrm{rxn}}{ }=-1072 \mathrm{~kJ} / \mathrm{mol}$

1. Write the enthalpy of formation reaction for $\mathrm{CS}_{2}$

2. Add the given rxns so that the result is the desired rxn.
$C$ (graphite) $)+Q_{2}(g) \longrightarrow C Q_{2}(g) \quad \Delta H_{r x n}^{\circ}=-393.5 \mathrm{~kJ} / \mathrm{mol}$
2 S (rhombic) $+2 Q_{2}(g) \longrightarrow 2 \mathrm{SO}_{2}(\mathrm{~g}) \quad \Delta H_{\mathrm{rx}}^{\circ}=-296.1 \mathrm{~kJ} / \mathrm{mol} \times 2$ $+\mathrm{CO}_{2}(g)+2 \mathrm{SO}_{2}(g)-\mathrm{CS}_{2}(I)+3 \mathrm{O}_{2}(g) \quad \Delta H^{\circ} \quad=+1072 \mathrm{~kJ} / \mathrm{mol}$

$$
\begin{aligned}
& \mathrm{C} \text { (graphite) }+2 \mathrm{~S} \text { (rhombic) } \longrightarrow \mathrm{CS}_{2}(I) \\
& \Delta H_{\mathrm{rxr}}^{\circ}=-393.5+(2 \mathrm{x}-296.1)+1072=86.3 \mathrm{~kJ} / \mathrm{mol}
\end{aligned}
$$

Benzene $\left(\mathrm{C}_{6} \mathrm{H}_{6}\right)$ burns in air to produce carbon dioxide and liquid water. How much heat is released per mole of benzene combusted? The standard enthalpy of formation of benzene is 49.04 kJ/mol.

$$
\begin{gathered}
2 \mathrm{C}_{6} \mathrm{H}_{6}(I)+15 \mathrm{O}_{2}(g) \longrightarrow 12 \mathrm{CO}_{2}(g)+6 \mathrm{H}_{2} \mathrm{O}(I) \\
\Delta H_{\mathrm{rxn}}^{\circ}=\Sigma n \Delta H_{\mathrm{f}}^{\circ}(\text { products })-\Sigma m \Delta H_{\mathrm{f}}^{\circ}(\text { reactants }) \\
\Delta H_{\mathrm{rxn}}^{\circ}=\left[12 \Delta H_{\mathrm{f}}^{\circ}\left(\mathrm{CO}_{2}\right)+6 \Delta H_{\mathrm{f}}^{\circ}\left(\mathrm{H}_{2} \mathrm{O}\right)\right]-\left[2 \Delta H_{\mathrm{f}}^{\circ}\left(\mathrm{C}_{6} \mathrm{H}_{6}\right)\right] \\
\Delta H_{\mathrm{rxn}}^{\circ}=[12 \mathrm{x}-393.5+6 \mathrm{x}-187.6]-[2 \mathrm{x} 49.04]=-5946 \mathrm{~kJ} \\
\frac{-5946 \mathrm{~kJ}}{2 \mathrm{~mol}}=-2973 \mathrm{~kJ} / \mathrm{mol} \mathrm{C}_{6} \mathrm{H}_{6}
\end{gathered}
$$

